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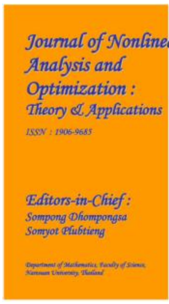

SOME PROPERTIES OF KNOT HYPERTPATH OF HYPERGRAPHS AND INCIDENCE GRAPHS

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Abstract
Knot hyperpaths, a fundamental concept in hypergraph theory, exhibit unique properties that contribute to the understanding of connectivity and traversal patterns in hypergraphs. The article delves into some characteriza- tion of knot hyperpaths in hypergraphs and their significance in its incidence graph. Moreover, the article motivates a technique for enumerating paths in hypergraphs.

Keywords: Graphs; Hypergraphs; Incidence graph; Path. 2020 Mathematics Subject Classification**:** 05C65, 05C38, 05C40.

1 Introduction

Many real-world problems, whether they have their roots in artificial or natural phenomena, involve a network-like structure. Graph theory, the architect of struc- tural connections and relationships, simplifies the intricate network through figures dealing with 'points and lines' intuitively. The abstract nature of graphs facilitates various aspects of the analysing process and improves visualization of complex, intri- cate relationships among the objects that other visualization methods cannot [1, 2]. However, graphs as a tool for modelling are extensively used in various fields and actively studied by researchers; they only support pairwise relationships between the vertices [3]. By contrast, there are many real-world problems where interactions among the objects may not be pairwise [4]. For example, in the case of the spread of infectious disease, where the rate of infection may be high and transmission of disease could occur in some groups of people, in the process of decision-making, such as in a grading system, where an object may have a higher rate of acceptance by multiple users, etc. It is to be noted that the complexity encountered by such highly heterogeneous systems, where the interactions of the objects are not necessarily pairwise, may be modelled by hypergraphs [5].

Claude Jacques Berge introduced hypergraph in 1973, as a means to generalise the approaches of graphs preserving the multi-adic relationship of the objects [6]. Thus, hypergraph becomes more natural models for studying the complexity of multi-adic relationships, which are currently attracting a lot of interest [7]. Essen- tially, the application of hypergraphs is not confined to specific domains. In fact, it transcends specific fields of study and making it a fundamental component in interdisciplinary research work as well. It is to be noted that, many fundamental concepts like path, cycle, tree, connectivity, coloring problems, etc. in graphs have been generalized to hypergraph theory, and their different well-known properties have been studied [8, 9, 10, 11]. Particularly, the concept of path in hypergraph has been studied in different ways and has gain lots of attention in past as it rep- resents the foundation of many research works. For instance, the problem related to shortest path like K-shortest hyperpaths in a directed hypergraph [12], connec- tivity measures using the concept of flags and pseudo path [13], eulerian circuit in hypergraphs using different sub edge in walk [14] and many more explicitly. Re- cently, Rahman et al. [15], introduced the concept of the knot and knot hyperpath of a hypergraph. A knot tend to be a fundamental concept in the hypergraph as it physically signifies the set of non-empty subset of some intersecting hyperedges. Moreover, using the concept of knot and strength of knot cut knot has been studied in [16] and studied the notion of separability in hypergraphs.

At the same time, it is geometrically evident that the intensity of connectiv- ity in hypergraphs varies across different regions with respect to both vertex and hyperedge connections. Thus, it is expedient to have a measure of the connect- edness of the vertices and hyperedges in a hypergraph. One way of doing this is by transforming the problem of hypergraphs into the problem of graphs. Thus, by leverging the inherent unique properties of the graph, the problems associated with great complexity can be solved or optimised with ease. Therefore, it is of interest to consider the notion of the incidence graph of the hypergraph. Since incidence graph representation of a hypergraph not only preserves the multi-adic relationships but also helps in visualization of the hypergaph as well into simpler form.

The subsequent sections of the article are structured as follows: Section 2 includes preliminary definitions with suitable examples. In Section 3, the key concept of knot hyperwalk in hypergraphs introduced in Section 2 has been characterised in its incidence graphs. Some results are also dedicated to hypertree and establish some examples, followed by a brief discussion on future perspectives with a conclusion section that ends our study.

2 Basic definitions and notations

As given by Berge [17], a hypergraph is a pair $H = (V, E)$, defined on a finite set of elements $V = \{v_1, v_2, \cdots, v_n\}$ called a vertex set, where the elements are called vertices, and $E = \{e_1, e_2, \cdots e_m\}$ is a collection of non-empty subsets of V called hyperedges or simply edges of the hypergraph. The cardinality of the vertex set V, that is, $|V|$, and the cardinality of the edge set E, that is, $|E|$, are known as the order and size of the hypergraph, respectively. The degree of a vertex $v \in V$ of a hypergraph $H = (V, E)$ is the number of hyperedges containing the vertex v, whereas the degree of a hyperedge $e \in E$ is the number of vertices contained in e. In a hypergraph, if the family of hyperedges satisfies $i \neq j \Longleftrightarrow e_i \neq e_j$, we say that H is without repeated hyperedge, and if $e_i \subset e_j \Rightarrow i = j$, then we call H a simple hypergraph. In this paper, all hypergraphs are considered simple hypergraphs, and the occurrence of repeated hyperedges does not arise. For more general information on hypergraphs and graphs, readers may refer to [17] and [18], respectively.

Definition 2.1. [15] A knot K in a hypergraph $H = (V, E)$ is a non-empty subset of some intersecting hyperedges, denoted by $K \subseteq \cap e_i$, where $e \in E$, $i = 1, 2, \cdots, k$ and $k \geq 2$. In particular, if $K = \cap e_i$ for all $e_i \in E$, then K is called an entire knot.

Definition 2.2. Let $H = (V, E)$ be a hypergraph and $v_1, v_n \in V$. A knot hyperwalk joining v_1, $v_n \in V$ in H is an alternative sequence of knots and hyperedges of the form

$W \equiv \{v_1\}e_1K_1e_2K_2e_3 \cdots e_{n-1}K_{n-1}e_n\{v_n\}$

satisfying the following properties:

1. $v_1 \in e_1$ and $v_n \in e_n$,
2. $K_i \subseteq (e_i \cap e_{i+1})$ are knots and $K_i \neq K_{i+1}$, where $i = 1, 2, \cdots, n-1$, and
3. $e_i \in E$ are hyperedges (not necessarily distinct).

In W, as defined above in Definition 2.2, if each hyperedge is distinct, then the knot hyperwalk is said to be a knot-hypertrail. Moreover, a knot hyperwalk in which neither knots nor hyperedges are repeated is called a knot hyperpath [15]. It is to be noted that if $K_i = e_i \cap e_{i+1}$ for all $i = 1, 2, \cdots, n-1$, then the knot-hyperwalk is called the entire knot-hyperwalk; analogously, the entire knot-hypertrail and the entire knot hyperpath can be defined. Moreover, if the end vertices coincide, then the knot hyperwalk (trail) is said to be a closed knot hyperwalk (tour), respectively.

Definition 2.3. [16] A hypergraph $H = (V, E)$ is said to be a cyclic hypergraph if there exist an entire knot hypercycle

$\{v_1\}e_1K_1e_2K_2e_3 \cdots e_{n-1}K_{n-1}e_1\{v_n\}$

such that $e_1, e_2, \cdots, e_{n-1}$ are the only hyperedges of the hypergraph H.

As an illustration of the above definitions, the following example has been dis- cussed:

Example 2.1. Let $H = (V, E)$ be a hypergraph with the vertex set $V = \{v_i | i = 1, 2, \cdots, 20\}$ and hyperedges $E = \{e_i : i = 1, 2 \cdots, 8\}$, where $e_1 = \{v_1, v_2, v_3, v_4\}$, $e_2 = \{v_4, v_5, v_6, v_7, \}$, $e_3 = \{v_7, v_8, v_8, v_9, v_{10}, v_{11}\}$, $e_4 = \{v_{11}, v_{12}, v_{13}, v_{14}\}$, $e_5 = \{v_{14}, v_{15}, v_{16}, v_{17}\}$, $e_6 = \{v_{16}, v_{17}, v_{18}\}$, $e_7 = \{v_3, v_4, v_{17}, v_{19}, v_{20}\}$, and $e_8 = \{v_{10}, v_{11}, v_{20}\}$.
The hypergraph $H = (V, E)$ is shown in Figure 1. All the possible knots and entire knots can be listed as follows: Let $K_1 = \{v_4\} = e_1 \cap e_2 \cap e_7 = e_1 \cap e_2$,

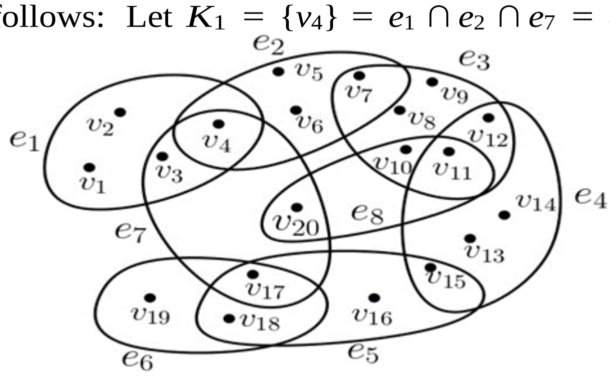


Figure 1: Hypergraph $H = (V, E)$ representing Example 2.1.

$K_2 = \{v_3\} \subset e_1 \cap e_7$, $K_3 = \{v_3, v_4\} = e_1 \cap e_7$,
$K_4 = \{v_7\} = e_2 \cap e_3$, $K_5 = \{v_{10}\} \subset e_8 \cap e_3$, $K_6 = \{v_{10}, v_{11}\} = e_8 \cap e_3$,
$K_7 = \{v_{11}\} = e_3 \cap e_8 \cap e_4$, $K_8 = \{v_{12}\} \subset e_3 \cap e_4$, $K_9 = \{v_{11}, v_{12}\} = e_3 \cap e_4$, $K_{10} = \{v_{15}\} = e_4 \cap e_5$, $K_{11} = \{v_{17}\} = e_5 \cap e_6 \cap e_7$, $K_{12} = \{v_{18}\} \subset e_5 \cap e_6$,
$K_{13} = \{v_{17}, v_{18}\} = e_5 \cap e_6$ and $K_{14} = \{v_{20}\} = e_7 \cap e_8$. Observe that,

1. $W \equiv \{v_{20}\}e_7K_1e_2K_4e_3K_5e_8K_{14}e_7K_{11}e_6\{v_{19}\}$ is a knot hyperwalk, but not a knot hypertrail, because the hyperedge e_7 occurs twice.
2. $W' \equiv \{v_{20}\}e_8K_5e_3K_4e_2K_1e_7K_{11}e_6\{v_{19}\}$ is a knot hypertrail. Moreover W' is a knot hyperpath joining v_{20} and v_{19}.
3. $P \equiv \{v_{16}\}e_5K_{10}e_4K_7e_8K_{14}e_7K_{11}e_5\{v_{18}\}$ is a knot hypercycle. However, the hypergraph $H = (V, E)$ is not cyclic (see Definition 2.3).

Definition 2.4. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs. Two knot hyperpaths
And

$$P \equiv \{v_1\}e_1K_1e_2 \cdots e_{n-1}K_{n-1}e_n\{v_n\}$$
$$P' \equiv \{v'_1\}e'_1K'_1e'_2 \ldots\ldots e'_{n-1}K'_{n-1}e'_n\{v'_n\}$$

of the same length in H_1 and H_2, respectively, are said to be equivalent if there exists a mapping f from V_1 to V_2 satisfying the following conditions:
i. $e_i \cap f^{-1}(e'_i) \neq \emptyset$ for all $i = 1, 2, \cdots n$.
ii. $K_i \cap f^{-1}(K'_i) \neq \emptyset$ for all $i = 1, 2, \cdots n - 1$.
The definition is illustrated in the following example
Example 2.2. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs, where $V_1 = \{v_1, v_2, \cdots v_{14}\}$, $E_1 = \{e_1, e_2, \cdots, e_5\}$ such that $e_1 = \{v_1, v_2, v_3, v_{11}\}$, $e_2 = \{v_2, v_6, v_{10}, v_{11}\}$, $e_3 = \{v_2, v_4, v_6, v_9, v_{11}, v_{13}\}$, $e_4 = \{v_4, v_7, v_8, v_9, v_{12}\}$, $e_5 = \{v_4, v_5 v_9, v_{12}, v_{14}\}$, and $V_2 = \{v'_1, v_2, \cdots, v'_{10}\}$,
$e'_1 = \{v'_1, v'_2, v'_3\}$, e' $e'_2 = \{v'_2, v'_3, v'_7, v'_9\}$, $e'_3 = \{v'_4, v'_7, v'_{10}\}$, $e'_4 = \{v'_4, v'_6, v'_8\}$, and $e'_5 = \{v'_4, v'_5, v'_8, v'_{10}\}$. The graphical representations for H_1 and H_2 are shown below in Figure 2(a) and Figure 2(b), respectively. We consider two knot hyperpaths,
$P \equiv \{v_1\}e_1\{v_2, v_{11}\}e_2\{v_6\}e_3\{v_4, v_9\}e_4\{v_7\}$
And
$P \equiv \{v'_1\}e'_1\{v'_2\}e'_2\{v'_7\}e'_3\{v'_4\}e'_4\{v'_6\}$
of length 4 of H_1 and H_2, respectively. Define $f : V_1 \longrightarrow V_2$ by $f(v_1) = v_1$, $f(v_2) = v'_3$, $f(v_3) = v'_2$, $f(v_4) = v'_4$, $f(v_5) = v'_5$, $f(v_6) = v'_7$, $f(v_7) = v'_6$, $f(v_8) = v'_8$, $f(v_9) = v'_{10}$, $f(v_{10}) = v'_9$, $f(v_{11}) = v'_2$, $f(v_{12}) = v'_8$, $f(v_{13}) = v'_{10}$, and $f(v_{14}) = v'_5$. Then, $f^{-1}(v'_1) = v_1$, $f^{-1}(v'_2) = \{v_3, v_{11}\}$, $f^{-1}(v'_3) = v_2$, $f^{-1}(v'_4) = v_4$, $f^{-1}(v'_5) = \{v_5, v_{14}\}$, f

$^{-1}(v'_6) = v_7$, $f^{-1}(v'_7) = v_6$,$_1$ $f^{-1}(v'_8) = \{v_8, v_{12}\}$ $f^{-1}(v'_9) = v_{10}$, $f^{-1}(v'_{10}) = \{v_9, v_{13}\}$, and $f^{-1}(v'_1) = v_1$. Thus, P and P' satisfies the two conditions $e_i \cap f^{-1}(e'_1{}') \neq \emptyset$ and $K_i \cap f^{-1}(K'_1{}') \neq \emptyset$. Hence P and P' are equivalent hyperapths.

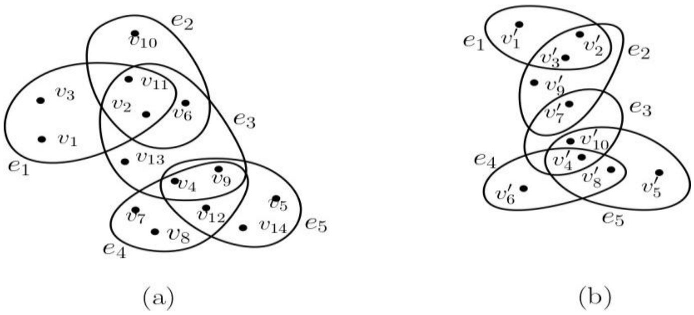


Figure 2: Hypergraphs $H_1 = (V_1, E_2)$ and $H_2 = (V_2, E_2)$ with equivalent hyperpaths.

Remark 2.1. If the mapping f in Definition 2.4 is from V to V of a hypergraph, then the definition coincide with Definition 6 of [15].

Definition 2.5. [13] Let $H = (V, E)$ be a hypergraph. The incidence graph I_H of H is the graph $I_H = (V_G, E_G)$ with $V_G = V \cup E$ and $E_G = \{ve : v \in V, e \in E\}$.

For our convenience, we denote a walk in incidence graph I_H of a hypergraph H as a sequence of v–vertices and edges of the form

$v_1(v_1e_1)e_1(v_2e_1)v_2(v_2e_2)e_2 \cdots v_{n-1}(v_{n-1}e_n)e_n(e_nv_n)v_n$

joining v_1 and v_n in I_H, where (v_ie_i) are edges in I_H. For instance, consider

$W = \{v_2\}(v_2e_1)\{e_1\}(e_1v_3)\{v_3\}(v_3e_2)\{e_2\}(e_2v_7)\{v_7\}$

is a path joining v_2 and v_7 in I_H of H in Figure 3.

Example 2.3. The incidence graph I_H of $H = (V, E)$ in Example 2.2 as shown in Figure 2(a) can be represented as shown below in Figure 3.

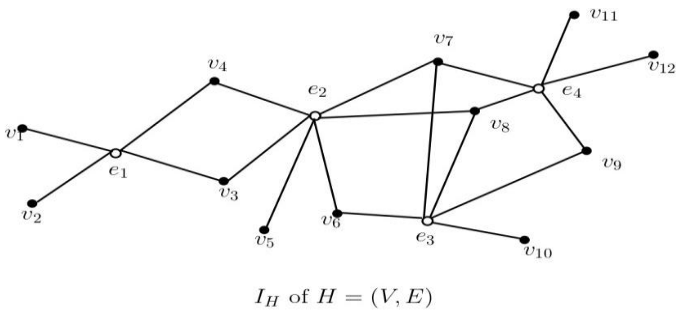


Figure 3: Incidence graph I_H of $H = (V, E)$ in Example 2.2, Figure 2(a).

3 Characterization of Knot hyperpaths in I_H

In this section of the paper, some characterization of knot hyperwalk (path) of hypergraphs introduced in Section 2 with respect to its incidence graph has been done.

Lemma 3.1. *Let $H = (V, E)$ be a connected hypergraph, and let I_H be the incidence graph of H. Then, for every knot hyperpath P in H, there exists a path P' in I_H.*

Proof. Let $P \equiv \{v_1\}e_1K_1 \cdots e_{n-1}K_{n-1}e_n\{v_n\}$ be a knot hyperpath in H, and let I_H be the incidence graph of H. By Definition 2.2, $v_1 \in e_1$ and $v_n \in e_n$; therefore, (v_1e_1) and (v_ne_n) are edges in I_H joining v_1 with e_1 and v_n with e_n, respectively. Since for each i, K_i is a non-empty subset of $e_i \cap e_{i+1}$, we can choose a vertex $v_{i+1} \in K_i$. Therefore, $v_{i+1} \in e_i$ and $v_{i+1} \in e_{i+1}$. It follows that v_{i+1} incident in e_i and e_{i+1}. Thus, $(v_{i+1}e_i)$ and $(v_{i+1}e_{i+1})$ are edges in I_H. Thus, we obtain an alternating sequence $P \equiv v_1(v_1e_1)e_1(v_2e_1)v_2(v_2e_2)e_2 \cdots e_n(e_nv_n)v_n$ of vertices and edges in I_H corresponding to the knot hyperpath P in H (see Figure 4). □

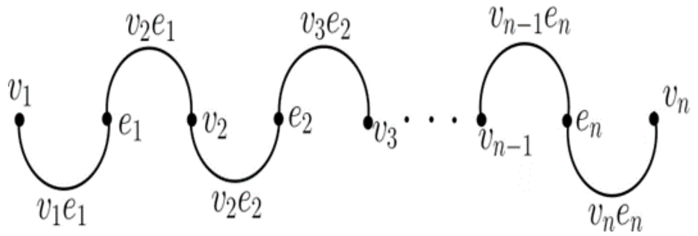


Figure 4: Schematic diagram of path P in incidence graph I_H for Lemma 3.1.

The converse is not true. However, if the end vertices of any path in I_H are $v-$ vertices, then the converse is also true.

Proposition 3.2. *Let $H = (V, E)$ be a hypergraph, and let I_H be the incidence graph of H. Then, for any path P' in I_H, whose ending vertices are $v-$vertices, a knot hyperpath corresponding to P' can be generated in H.*

Proof. Any path in I_H must be in the following form:

$P' = v_1(v_1e_1)e_1(v_2e_1)v_2(v_2e_2)e_2 \cdots e_{n-1}(v_ne_{n-1})v_n(v_ne_n)e_n(v_{n+1}e_n)v_{n+1}$.

Since $v_1 \in e_1$ and $v_2 \in e_1 \cap e_2$, choose $\{v_2\} \subseteq K_1 \subseteq e_1 \cap e_2 \neq \emptyset$. Since $v_2 \neq v_3$ and $v_3 \in e_2 \cap e_3$, choose $\{v_3\} \subseteq K_2 \subseteq e_2 \cap e_3 - K_1$. This process may be continued. After (n-2)th steps, we can choose

$\{v_n\} \subseteq K_{n-1} \subseteq e_{n-1} \cap e_n - \bigcup_{i=1}^{n-2} K_i$

Since the last vertex v_{n+1} in I_H is adjacent to e_n, we have $v_{n+1} \in e_n$ in H. Hence we obtain a sequence of knots and hyperedges

$P \equiv \{v_1\}e_1K_1e_2K_2e_3 \cdots K_{n-1}e_n\{v_{n+1}\}$

such that $K_i \subseteq e_i \cap e_{i+1} - \bigcup_{j=1}^{i-1} K_j$ for each $i = 2, 3, \cdots n-1$. Hence, P is a knot hyperpath in H. □

Proposition 3.3. *Let $H = (V, E)$ be a hypergraph, and let I_H be its incidence graph. For every knot hyperpath P of length l in H, there exists a path P' of length $2l$ in I_H.*

Proof. We proceed with the proof by induction on l. If $l = 1$, that is, $P \equiv \{v_1\}e_1\{v_2\}$ a knot hyperpath of length 1 in H, then by Lemma 3.1, the corresponding path of P in I_H must be of the form $P' = v_1(v_1e_1)e_1(v_2e_1)v_2$. Since (v_1e_1) and (v_2e_2) are two edges in P'_1. Thus, $P'_1 = v_1(v_1e_1)e_1(v_2e_2)v_2$ is a path of 1
in I_H. We assume 1 1 ˋ

length 2

that the result is true for $m < l$. If

$P1 \equiv \{v1\}e1K1e2K2 \cdots em-1Km-1emKmem+1\{vm+2\}$

is a knot hyperpath of length $m + 1$ in H. Then P_1 can be written as the union of two hyperpaths $P_1 = P' \cup P''$, where

1 1

$P' \equiv \{v_1\}e_1K_1e2 \cdots e_{m-1}K_{m-1}e_m\{v_{m+1}\}$

And 1

$P'' \equiv \{vm+1\}em+1\{vm+2\}$ 1

with $v_{m+1} \in K_m \subseteq e_m \cap e_{m+1}$. Since the lengths of the knot hyperpaths P'_1 and $P^n{}_1$ are m and 1, respectively, the corresponding paths P'_2 and $P^n{}_2$ in I_H must be of the form

and 2

$P' = v_1(v_1e_1)e_1(v_2e_1)v_2(v_2e_3)e_3 \cdots (v_me_m)e_m(e_mv_{m+1})v_{m+1}$

$P'' = vm+1(vm+1em+1)em+1(em+1vm+2)vm+2$ 2

with lengths $2m$ and 2, respectively. Since $v_{m+1} \in K_m \subseteq e_m \cap e_{m+1}$, the path $P_2 = P'_2 \cup P'_2$ in I_H is of length $2(m + 1)$. □

Theorem 3.4. *If $P \equiv \{v_1\}e_1K_1 \cdots e_{k-1}K_{n-1}e_k\{v_n\}$ is an entire knot hyperpath in a hypergraph $H = (V, E)$, then $|K_1| \times |K_2| \cdots |K_{n-1}|$ number of paths can be constructed in I_H corresponding to P joining v_1 and v_n.*

Proof. Let $P \equiv \{v_1\}e_1K_1 \cdots e_{n-1}K_{n-1}e_n\{v_n\}$ be an entire knot hyperpath in a hypergraph H.

Let $|K_i| = r_i$, where $i = 1, 2, ..., n-1$. For each i, any vertex v in K_i incident in both the hyperedges e_i and e_{i+1}. It follows that (ve_i) and (ve_{i+1}) are edges in I_H joining v with e_1 and e_2, respectively. Since $|K_i| = r_i$, by Lemma 3.1, e_i and e_{i+1} can be joined by r_i ways in I_H. For each such links, we shall have a path in I_H corresponding to P. For r_1 different vertices in K_1, r_1 paths can be constructed in I_H corresponding to P, and, independently, for r_2 different vertices in K_2, r_2 paths can be constructed in I_H corresponding to P. Together with $r_1 \times r_2$ paths can be constructed in I_H corresponding to P.

This further continues till the last entire knot and since the last knot is K_{n-1}, the process terminates after $(k-1)$th steps. Thus continuing successively in the similar fashion for each such entire knot K_i, $i = 3, 3, ..., n-1$ in P of H, $r_1 \times r_2 \times \cdots r_{n-1}$ paths can be constructed in I_H corresponding to P. □

Corollary 3.5. *If $P \equiv \{v_1\}e_1K_1 \cdots e_{k-1}K_{n-1}e_k\{v_n\}$ is an entire knot hyperpath in a hypergraph $H = (V, E)$ with $|K_i| = r$ for some positive integer, then r^{k-1} number of paths can be constructed in I_H corresponding to P joining v_1 and v_n.*

Proposition 3.6. *Let $H = (V, E)$ be a hypergraph. If the incidence graph I_H of H is a tree, then H is a hypertree with every knot having cardinality 1. But the converse is not true.*

Proof. Let us assume that the incidence graph I_H of a hypergraph H is a tree. Since every tree is a connected graph, the hypergraph is also connected, and vice versa.
Firstly, we show that H is a hypertree. Later, we claim that every entire knot of H has cardinality 1.

To show that H is a hypertree, it will be sufficient if we show that every knot K in H is of strength greater than or equal to 1 ([16] Theorem 3.8). We proceed with the proof by using the method of contradiction. If possible, let us assume that there exists a knot K in H such that $St(K) = 0$; this implies that the removal of the knot K from H does not disconnect the hypergraph. Thus, there exist at least two distinct entire knot hyperpaths

$P_1 \equiv \{u\}e_1K_1e_2K_2 \cdots K_{n-1}e_n\{v\}$

and

$P_2 = \{u\}e'_1 K'_1e'_2K'_2 \cdots K'_{n-1}e'_n\{v\}$

joining two distinct vertices u and v in H. Then, by Lemma 3.1, there exist two distinct paths

$P_1 = u(ue_1)e_1(v_2e_1)v_2(v_2e_2)e_2 \cdots v_{n-1}(v_{n-1}e_n)e_n(ve_n)v$

and

$P'_1 = u(ue'_1)e'_1(v'_2e'_1)v'_2(v'_2e'_2)e'_2 \cdots v'_{n-1}(v'_{n-1}e'_n)e'_n(ve'_n)v.$

corresponding to P_1 and P_2 joining u and v in I_H, which is a contradiction to the fact that there exists a unique path joining any two vertices in a tree. Hence, H is a hypertree. Since $K_i = (e_i \cap e_{i+1}) K_j$ is an entire knot. Suppose $K \subseteq e \cap e'$ is a knot in H with $|K| > 1$. Let $u, v \in K_i$ be two distinct vertices. Since both u and v are adjacent to each intersecting hyperedge e and e' in H, a cycle

$u(ue)e(ve)v(ve')e'(ue')u$

can be formed in I_H (Figure 5). This shows that I_H is not a tree, a contradiction. Hence, $|K| = 1$ for every entire knot K in H. The converse does not hold true; for

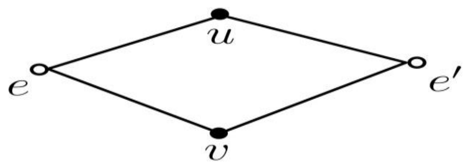
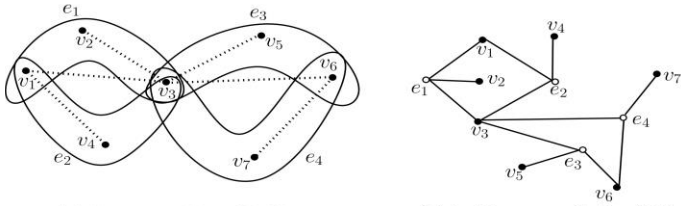


Figure 5: Cycle representation in I_H.

instance, consider the hypergraph $H = (V, E)$, where $V = \{v_i : i = 1, 2, 3, \cdots, 7\}$ and $E = \{e_1, e_2, e_3, e_4\}$ such that $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_1, v_3, v_4\}$, $e_3 = \{v_3, v_5, v_6\}$, and $e_4 = \{v_3, v_6, v_7\}$ (Figure 6(a)). Clearly, H is a hypertree with each knot's cardinality 1, but its incidence graph I_H is not a tree. (Figure 6(b)). □



(a) Hypertree $H = (V, E)$. (b) Incidence graph I_H of H.

Figure 6: A hypertree and its incidence graph.

4 Discussions and Conclusion

By leveraging the properties of graphs, we have characterized the concept of knot hyperpaths introduced by Rahman et al. [15] in its incidence graph. With the concept of incidence graphs, which offer a powerful representation of relationships, this study provides a nuanced understanding of knot hyperpaths, connectivity, and traversal problems in hypergraphs. The findings contribute not only to the theoretical aspects of the hypergraph theory, but they could also hold practical implications for optimizing network pathways in various applications in future research. However, such development is too early to declare at this moment, but optimistic.

This research contributes to the broader field of hypergraph theory, offering insights into the interplay between knot hyperpaths and incidence graphs. In this article, we are able to charterize the knot hyperpath in a hypergraph in its incidence graph. In our opinion, the method could be an interesting topic to investigate under the properties of Eulerian and Hamiltonian hypergraphs too. That is, such a study could be beneficial for computing the complexities associated with enumerating the path-related problems in future studies.

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